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Kolmogorov-Landau Functions

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A form of the Kolmogorov-Landau inequality that can be applied to any perfect set is suggested.

1. Introduction

The classical Kolmogorov inequality

$$\|f^{(q)}\| \leq C_{rq}\, \|f\|^{1-q/r}\, \|f^{(r)}\|^{q/r}$$

was proved by Kolmogorov in [11] for $f \in W^r_\infty(\mathbb{R})$. Here $\|\cdot\|$ means $\|\cdot\|_{L_\infty(\mathbb{R})}$ and the sharp constant $C_{rq} = K_{r-q}\, K_r^{-1+q/r}$ is given in terms of the Favard constants $K_p := \frac{4}{\pi}\sum_{n=0}^{\infty}\big[\frac{(-1)^n}{2n+1}\big]^{p+1}$. For the definition of the Sobolev space $W^r_\infty(\mathbb{R})$ see e.g. [4, Ch. 1.5]. The first results of this type (the case $q = 1$, $r = 2$) were obtained by Hadamard in [9] and by Landau in [12] (for $\|\cdot\|_{L_\infty(\mathbb{R}_+)}$), who initiated this class of extremal problems. Shilov [14] found the exact constants in the case $r \leq 4$, $q \leq r$ and predicted the class of extremal functions (Euler's splines) for the inequality in the general case. His conjecture was confirmed by A. N. Kolmogorov. After that, inequalities of Kolmogorov type, especially versions for a finite interval, attracted attention of many authors. For a contribution by Bojanov, see [1, 2, 3].

Let us consider the additive form of the Kolmogorov inequality

$$\|f^{(q)}\| \leq t^{q/(r-q)}\, \|f\| + \frac{C}{t}\, \|f^{(r)}\|, \qquad \forall t > 0. \tag{1}$$

In order to get the sharp inequality (1), one has to take C with

$$C^q = C^r_{rq}\, q^q\, (r-q)^{r-q}\, r^{-r}, \tag{2}$$

as is easy to check.

The exponent $\frac{q}{r-q}$ of t in (1) cannot be reduced at the expense of C. Indeed, let φ satisfy the condition $\underline{\lim}_{t\to\infty}\, \varphi(t)\, t^{-q/(r-q)} = 0$. Then for any fixed C the function $f(x) = \sin(A_n x)$ with a suitable A_n and n large enough violates the inequality

$$\|f^{(q)}\| \le \varphi(t)\,||f|| + \frac{C}{t}\,\|f^{(r)}\| \tag{3}$$

for some t. Therefore the function $\varphi_{q,r,\mathbb{R}}(t) = t^{q/(r-q)}$ gives the minimal possible growth of φ in (3).

Our aim is to generalize (3) so that in its new form it can be applied to classes $W^r_\infty(F)$ for closed subsets of $\mathbb{R}$ or for more general function spaces.

2. Kolmogorov-Landau Functions

Let F be a compact subset of $\mathbb{R}$. Then, for any function of the set $\Phi := \{\varphi : \varphi \in C([0,\infty))$ with $\varphi \uparrow$ and $\varphi(0) = 0\}$ and any constant C one can easily find a polynomial f that destroys (3). We want to consider interpolating inequalities in classes of functions containing, at least, polynomials, so let us replace $\|f^{(r)}\|$ by $\|f\|_r := \max_{0\le k\le r} \|f^{(k)}\|$.

Thus we arrive at the following definition: given a perfect set $F \subset \mathbb{R}$ and $1 \le q \le r-1$, we say that a function $\varphi \in \Phi$ is a (q,r)-Kolmogorov-Landau function for F (written $\varphi \in \Phi_{q,r,F}$) if there exists a constant C such that the inequality

$$\|f^{(q)}\| \le \varphi(t)\,||f|| + \frac{C}{t}\,\|f\|_r \tag{4}$$

holds for each $t > 0$ and $f \in W^r_\infty(F)$.

This definition can be applied only to perfect subsets of $\mathbb{R}$ since otherwise f with $\|f\| = 0$, $\|f^{(q)}\| = 1$ violates (4) for each φ.

Clearly, $\Phi_{q+1,r,F} \subset \Phi_{q,r,F} \subset \Phi_{q,r+1,F}$.

Of course, each (q,r)-Kolmogorov-Landau function, even of optimal growth, is not unique and can be replaced by 0 on interval $[0, t_0]$ for any t_0 given beforehand.

Thus, given F, q, and r, we first get the problem of finding the minimal growth of φ in (4), and then, given φ, we get the problem of minimization of C in (4). Any function realizing the minimal growth will be denoted by $\varphi_{q,r,F}$.

3. Linear Topological Invariants

The condition (4) is closely related to interpolating linear topological invariants of D_1- or DN-type.

Let X be a Fréchet space with an increasing sequence of seminorms $(\|\cdot\|_p)_{p=0}^\infty$. Generalizing the class (d_1) of Dragilev [5], Zahariuta introduced in [19] the class D_1 of Fréchet spaces satisfying the condition

$$\exists p \in \mathbb{Z}_+ \;\forall q \in \mathbb{N}\; \exists r \in \mathbb{N},\; C > 0\; :\; \|\cdot\|_q^2 \le C\|\cdot\|_p\,\|\cdot\|_r.$$

Independently, Vogt [17] considered this as the Dominating Norm Property, and proved that the existence of a dominating norm characterizes (in the class of nuclear Fréchet spaces) the subspaces of the model space s of rapidly decreasing sequences.

As a generalization, the following linear topological invariants were suggested to classify spaces of infinitely differentiable functions:

$$DN_\varphi \ \ ([18,\ 16]) : \ \exists\, p\ \forall q\ \exists r,\ C>0\ :\ \|\cdot\|_q \le \varphi(t)\ \|\cdot\|_p + \frac{C}{t}\,\|\cdot\|_r, \quad \forall t>0$$

and

$$D_\varphi \ ([6]) : \ \exists\, p\ \forall q\ \exists r,\ m>0,\ C>0 :\ \|\cdot\|_q \le \varphi^m(t)\ \|\cdot\|_p + \frac{C}{t}\,\|\cdot\|_r, \quad \forall t>0.$$

Note that the main difference between (4) and (DN_φ), provided $\|\cdot\|_0$ is a dominating norm in $W^\infty_\infty(F)$ is that in our case the function φ essentially depends on the pair (q,r), whereas in (DN_φ) this function must be the same for all pairs $(q, r(q))$. In Examples 3 and 4 below we use arguments from [7], [8], and [6] with more careful control on the function φ.

4. Examples

Example 1. For $F=\mathbb{R}$, due to A. N. Kolmogorov, the function $\varphi_{q,r,\mathbb{R}}(t) = t^{q/(r-q)}$ and the constant C defined in (2) give the sharp inequality (4). Indeed, Kolmogorov's inequality is realized on the Euler splines (e_r) (see e.g. [13] or [4] for the definition of Euler's splines). Moreover, the spline e_r realizes (1) for all q with $1\le q\le r-1$. Since $\|e_r^{(k)}\|$ increases with k (see e.g. [13, L. 7]), we have $\|e_r\|_r = \|e_r^{(r)}\|$ and sharpness of (4). Thus, (4) is indeed a generalization of (1).

Also, Kolmogorov's inequality implies that $W^\infty_\infty(\mathbb{R})$ has the property (DN_φ) for $\varphi(t)=t^\varepsilon$ with any $\varepsilon>0$ given beforehand.

Example 2. Let $F=[0,1]$. Then, as above, $\varphi_{q,r,[0,1]}(t) = t^{q/(r-q)}$ gives the minimal growth (see e.g. [4, T. 2.5.6]), but the exact value of C for the sharp inequality (1) is not known.

In the case of infinitely connected set F we have to specify the class $W^r_\infty(F)$. We will consider differentiability in the Whitney sense, that is $f\in W^r_\infty(F)$ if there exists a function $\tilde f \in W^r_\infty(\mathbb{R})$ such that $\tilde f^{(k)} = f^{(k)}$ on F for $0\le k\le r$. Then a topology of a Banach space in $W^r_\infty(F)$ can be determined by the norm

$$|||\, f\,|||_r = \|f\|_r + \sup\Big\{\frac{|(R^r_y f)^{(k)}(x)|}{|x-y|^{r-k}} : x,y\in F,\ x\ne y,\ k=0,1,\dots,r\Big\},$$

where $R_y^r f(x)$ is the Taylor remainder. Thus, in the next example we replace $\|f\|_r$ in (4) by $|||\, f\,|||_r$.

In [8] two interpolating lemmas were used for functions from C^r and, correspondingly, $\mathcal{E}^r$ classes (Lemma 1 is well-known, see [4, Theorem 2.5.6]). The same proof can be applied to functions from W_∞^r class and to any closed set in the second case, so we get

Lemma 1. *Let I be any closed interval in $\mathbb{R}$ with length$(I) = \delta_0$ and let $0 \le q \le r$ be given. Then there exist two constants C_1, C_2 such that*

$$\forall f \in W_\infty^r(I)\ \forall \delta \in (0, \delta_0]\ \forall x \in I\ :\ |f^{(q)}(x)| \le C_1\, \delta^{-q} \|f\| + C_2\, \delta^{r-q} \|f\|_r.$$

Lemma 2. *Let $F \subset \mathbb{R}$ be a closed set containing $r+1$ distinct points $x_0, \ldots, x_r$ such that $x_0 < x_1 < \cdots x_r$ and $h := x_1 - x_0 \le x_2 - x_1 \le \cdots \le x_r - x_{r-1} =: H$. Let $f \in W_\infty^r(F)$, $0 \le q \le r$. Then*

$$|f^{(q)}(x_0)| \le C_3\, h^{-q} \|f\| + C_4\, H^{r-q} |||f|||_r,$$

where C_3 and C_4 depend only on k and r.

One can use a technique from [8] in order to construct a set F with any growth of $\varphi_{q,r,F}$, which is given beforehand.

Example 3. Suppose $1 \le q \le r-1$ are given and $\varphi \in \Phi$ satisfies the conditions

$$2^{2q(r-q)} t^q \le \varphi^{r-q}(t) \quad \text{for all } t > 0, \tag{5}$$

$$\forall C\ \exists t_0\ :\ C\varphi(t) \le \varphi(Ct) \quad \text{for } t \ge t_0. \tag{6}$$

Let us construct a set F such that $\varphi \in \Phi_{q,r,F}$, whereas $\varphi_1 \notin \Phi_{q,r,F}$ for each function φ_1 with

$$\varphi_1(Ct)/\varphi(t) \to 0 \quad \text{as } t \to \infty \ \text{ for any fixed } C. \tag{7}$$

Let us take $b_n = 2^{-n},\ l_n = \varphi^{-1/q}(2^{n(r-q)}),\ I_n = [b_n - l_n, b_n]$, and $F = \{0\} \cup \bigcup_{n=1}^\infty I_n$. The condition (5) gives $b_n - l_n < b_n$, so the set F is well defined. Moreover, this condition implies $h_n \ge 2^{-n-2}$, where h_n denotes the distance between I_n and I_{n+1}.

To show that $\varphi_1 \notin \Phi_{q,r,F}$ for φ_1 satisfying (7), let us consider

$$f(x) = \begin{cases} (x - b_n)^q/q!, & \text{for } x \in I_n, \\ 0, & \text{for } x \in F \setminus I_n. \end{cases}$$

Then $\|f\| = l_n^q/q!,\ \ \|f^{(q)}\| = 1$, and $|||\, f\,|||_r = 1 + h_n^{q-r}$. If $\varphi_1 \in \Phi_{q,r,F}$, then there would exists a constant C such that

$$1 \le \varphi_1(t) \frac{1}{q!}\, \varphi^{-1}(2^{n(r-q)}) + \frac{C}{t}\, 2^{(n+2)(r-q)}$$

for all $t > 0$. But this is impossible for $t = 2C\,2^{(n+2)(r-q)}$ and large n in view of (7).

To prove that, thanks to (6), $\varphi \in \Phi_{q,r,F}$, it is enough to find constants A, B such that the inequality

$$|f^{(q)}(x)| \le A\,\varphi(t)\|f\| + \frac{B}{t}\,|||f|||_r \tag{8}$$

holds for each $f \in W^r_\infty(F)$, $x \in F$, and $t > 0$.

Given large t (let $t > 2^{r(r-q)}$), fix n with

$$2^{qn} \le \varphi(t) < 2^{q(n+1)}. \tag{9}$$

We have $n \ge r+2$ because of (5).

For any fixed $f \in W^r_\infty(F)$ we consider separately the different locations of $x \in F$. Suppose first that $x \le b_n$. Let us take $x_1 = b_{n-1}, x_2 = b_{n-2}, \ldots, x_r = b_{n-r}$, and use Lemma 2 with $x_0 = x$. Here, $h \ge b_{n-1} - b_n = 2^{-n}$, $H = 2^{-(n-r+1)}$, so $|f^{(q)}(x)| \le C_3\,2^{nq}\|f\| + C_4 2^{-(r-q)(n-r+1)}\,|||f|||_r$. Combining (5) and (9) we get (8) with $A = C_3$ and $B = C_4\,2^{(r-q)(r-2)}$.

Suppose now that $x \in I_k$ with $k = n-1, n-2, \ldots, k_0$, where k_0 is such that

$$2^{(k_0-1)(r-q)} < t \le 2^{k_0(r-q)}.$$

Then we take $x_1 = b_{k-1}, \ldots, x_r = b_{k-r}$. Due to the choice of t we have $k_0 \ge r+1$ and $k - r \ge 1$. Therefore, now $h \ge 2^{-k}$, $H = 2^{-(k-r+1)}$ and as above, $h^{-q} \le 2^{(n-1)q} < \varphi(t)$ and $H^{r-q} \le 2^{-(r-q)(k_0-r+1)} \le 2^{(r-q)(r-1)}/t$. Thus we get (8) with $A = C_3$ and $B = C_4\,2^{(r-q)(r-1)}$.

It remains to consider the case $x \in I_k$ with $k < k_0$. Let us show that the length of I_k for such k is large enough in order to apply Lemma 1. We will use it with $\delta = l_{k_0-1}$:

$$|f^{(q)}(x)| \le C_1\,l_{k_0-1}^{-q}\|f\| + C_2\,l_{k_0-1}^{r-q}\|f\|_r.$$

Here, $l_{k_0-1}^{-q} = \varphi(2^{(k_0-1)(r-q)}) < \varphi(t)$ due to the choice of k_0. On the other hand, $l_{k_0-1}^{r-q} < 2^{q-r}/t$. Indeed, $2^{r-q}\,t \le 2^{2(r-q)}\,2^{(k_0-1)(r-q)} \le \varphi^{(r-q)/q}(2^{(k_0-1)(r-q)})$, by (5). The latter value is just $l_{k_0-1}^{q-r}$.

Gathering all the cases considered, we get (8) with $A = \max\{C_1, C_3\}$ and $B = \max\{C_2\,2^{q-r}, C_4\,2^{(r-q)(r-1)}\}$.

The set F from Example 3 is, in a sense, a one-dimensional analog of a sharp cusp on the plane. Clearly, for functions of two variables we have to take $\max\{\|f^{(q_1,q_2)}\| : q_1 + q_2 = q\}$ instead of $\|f^{(q)}\|$ on the left part of (4). On the other hand, since the set in the next example is Whitney regular, we can return to (4) in its initial with respect to $\|\cdot\|_r$ form.

Example 4. Let $\psi : [0,1] \to [0,1]$ be a nondecreasing function with the properties:

$$\psi(0) = 0;\ 0 < \psi(\tau) \le \tau \ \text{ for } \ 0 < \tau < \tau_0 < 1;\ \psi(\tau) = \psi(\tau_0) \ \text{ for } \ \tau_0 \le \tau \le 1.$$

Let $F_\psi = \{(x,y) : 0 \le x \le 1,\ |y| \le \psi(x)\}$.

Let us show that $\varphi \in \Phi_{q,r,F_\psi}$, provided that

$$\forall C\ \exists\ \tau_1\ :\ \psi^q(\tau)\cdot\varphi(\tau^{q-r}) \ge C \quad \text{for} \quad \tau < \tau_1, \tag{10}$$

whereas $\varphi_1 \notin \Phi_{q,r,F_\psi}$ for every function φ_1 such that for any fixed C

$$\psi^q(\tau)\,\varphi_1(C\tau^{q-r}) \to 0 \quad \text{as} \quad \tau \to 0. \tag{11}$$

If, in addition, $\varphi \in \Phi$ satisfies (6), then (10) can be relaxed to

$$\exists\tau_1\ :\ \psi^q(\tau)\,\varphi(\tau^{q-r}) \ge 1 \quad \text{for} \quad \tau < \tau_1.$$

If, in this case, we take $\psi(\tau) = \varphi^{-1/q}(\tau^{q-r})$, then (11) coincides with (7).

Suppose that φ_1 satisfies (11). Given $0 < \tau < 1$, let us take a non-increasing function $g_\tau \in C^\infty[0,1]$ such that

$$g_\tau(x) = \begin{cases} 1, & \text{for } 0 \le x \le \tau/2, \\ 0, & \text{for } x \ge \tau, \end{cases}$$

and $|g_\tau^{(k)}(x)| \le A_k\tau^{-k}$ for $1 \le k \le r$, $0 \le x \le 1$ with $A_1 \le A_2 \le \cdots \le A_r$. Then for $f_\tau(x,y) = y^q/q! \cdot g_\tau(x)$ we have $\|f_\tau\| \le \psi^q(\tau)/q!$, $\max\{\|f^{(q_1,q_2)}\| : q_1+q_2 = q\} \ge f^{(0,q)}(0,0) = 1$, and $\|f_\tau\|_r \le A_r\,\tau^{q-r}$. Therefore,

$$1 \le \varphi_1(t)\,\frac{\psi^q(\tau)}{q!} + C\,A_r\,\frac{\tau^{q-r}}{t},$$

which is impossible for $t = 2C\,A_r\tau^{q-r}$ in view of (11).

Suppose that φ satisfies (10). Let us show that there are constants A, B such that for all q_1, q_2 with $q_1 + q_2 = q$, $f \in W^r_\infty(F_\psi)$, and $(x,y) \in F_\psi$ we have for all $t > 0$

$$|f^{(q_1,q_2)}(x,y)| \le A\,\psi^{-q}(1/t)\|f\| + B\,t^{q-r}\,\|f\|_r. \tag{12}$$

By (10), this yields the desired result.

Given t, we define

$$F^t_\psi = \{(x,y) \in F_\psi : x \ge 1/t\}, \qquad \|f\|^{(t)} = \sup\{|f(x,y)| : (x,y) \in F^t_\psi\}.$$

Fix $f \in W^r_\infty(F_\psi)$ and $t > \max\{2, (1-\tau_0)^{-1}\}$. As in [6], we will realize the following plan:

1) an estimation of $\|f^{(0,k)}\|^{(t)}$ for $1 \le k \le q$;
2) an estimation of $|f^{(0,k)}(x,y)|$ for $x < 1/t$ in terms of $\|f^{(0,k)}\|^{(t)}$ and $\|f\|_r$;
3) a general bound of $|f^{(q_1,q_2)}(x,y)|$.

For 1) let us consider the function $h(y) = f(x,y)$, where $x \ge 1/t$ is fixed. Lemma 1, applied to h with $\delta = 2\,\psi(1/t)$ gives

$$\|f^{(0,k)}\|^{(t)} \le C_1\,(2\psi(1/t))^{-k}\|f\| + C_2\,(2\psi(1/t))^{r-k}\|f\|_r.$$

2) Let $0 \le x < 1/t$. We use the Taylor expansion around $(1/t, y)$ for the function $h(x) = f^{(0,k)}(x,y)$, where y with $|y| \le \psi(x)$ is fixed:

$$f^{(0,k)}(x,y) = \sum_{j=0}^{r-k} f^{(j,k)}(1/t,y)\,\frac{(x-1/t)^j}{j!} + R_{1/t}^{r-k}h(x).$$

Here, $|R_{1/t}^{r-k}h(x)| \le 2\|h\|_{r-k}\,t^{k-r} \le 2\|f\|_r\,t^{k-r}$. Now, in order to estimate $|f^{(j,k)}(1/t,y)|$ for $j > 0$ we apply Lemma 1 to $h \in W_\infty^{r-k}[1/t,1]$ at the point $x = 1/t$ with $\delta = 1/t$:

$$|f^{(j,k)}(1/t,y)| \le C_1\,t^j\,\|f^{(0,k)}\|^{(t)} + C_2\,t^{j+k-r}\,\|f^{(0,k)}\|_{r-k}.$$

From here,

$$|f^{(0,k)}(x,y)| \le \Big(1 + C_1\sum_{j=1}^{r-k}\frac{1}{j!}\Big)\|f^{(0,k)}\|^{(t)} + \Big(2 + C_2\sum_{j=1}^{r-k}\frac{1}{j!}\Big)t^{k-r}\,\|f\|_r.$$

3) To get (12), for any q_1, q_2 with $q_1 + q_2 = q$, we apply Lemma 1 to $h(x) = f^{(0,q_2)}(x,y)$ from the space $W_\infty^{r-q_2}[a,b]$. Here, $b - a \ge 1 - \tau_0$, so, due to the choice of t, one can take $\delta = 1/t$:

$$|f^{(q_1,q_2)}(x,y)| \le C_1\,t^{q_1}\,\|f^{(0,q_2)}\| + C_2\,t^{q_1+q_2-r}\,\|f\|_r.$$

Finally, a combination of the bounds from 1) and 2) together with $\psi(1/t) \le 1/t$ gives the desired conclusion.

Remark. Due to Stein ([15, Theorem 2], see also [4, Theorem 5.7.4]), the $\|\cdot\|_{L_p(\mathbb{R})}$-version of the Kolmogorov inequality holds as well for functions from $W_p^r(\mathbb{R})$ with $1 \le p < \infty$. It should be noted that the Hilbert case ($p = 2$) was considered by Hardy et al. in [10]. One can suggest the corresponding class $\Phi_{q,r,p,F}$ of functions satisfying the $\|\cdot\|_{L_p(\mathbb{R})}$-version of (4). It is interesting to compare $\Phi_{q,r,p,F}$ and our class $\Phi_{q,r,\infty,F}$ in the case when F is the closure of a domain where the Sobolev embedding theorem is not valid, for example as in Example 4 above with a small function ψ.

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